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On the Question of Decomposition of Multi-timescale Systems Dynamic Models

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***Abstract*—The problem of decomposition of a multi-timescale systems dynamic model is considered in this paper. The specific feature of these systems is the simultaneous presence of processes with essentially different speeds of free components of motions. The description model is represented by a singularly perturbed system of differential equations. The conditions are examined that allow reduction of the order of the system model: the checking of these conditions causes the greatest difficulty in practice. Research results are given as an iterative procedure requiring no special knowledge of singularly perturbed systems of differential equations.**

Keywords—multi-timescale system; singularly perturbed model; decomposition of a model; reduced model; associated system

I. Introduction

In problems of dynamic objects control, *multi-timescale systems* compose a separate class requiring special consideration. The essential feature of these systems is the simultaneous presence of processes with essentially different speeds of free components of motions. They are wide-spread, for instance, in electromechanical systems, models of which combine descriptions of fast-settling electrical processes and mechanical processes, this settling of which is determined by the inertia of the load [1]. Multi-timescale systems are particularly interesting because their models can be decomposed under certain conditions and thus the model order decreased.

The mathematical description of multi-timescale systems is usually expressed in the form of a singularly perturbed system of first-order differential equations with small parameters at the derivatives in some:

$$\begin{cases} \dfrac{d\vec{y}}{dt} = f\left(\vec{y},\vec{z},t\right), \\ \mu_i \dfrac{dz_i}{dt} = F_i\left(\vec{y},\vec{z},t\right), \end{cases} \quad i=\overline{1,m}, \tag{1}$$

where $\vec{y}=(y_1,\dots,y_n)^T$, $f=(f_1,\dots,f_n)^T$, $\vec{z}=(z_1,\dots,z_m)^T$; $z_i(t)$ and $F_i\left(\vec{y},\vec{z},t\right)$, $i=\overline{1,m}$, are scalar functions of their arguments; the parameter μ_{i+1} is of a higher order of magnitude than μ_i, so that, as they approach zero, $\frac{\mu_{i+1}}{\mu_i}\to 0$; $j=\overline{1,m}$; $n+m=s$ is the total order of the model.

Functions $f\left(\vec{y},\vec{z},t\right)$ and $F_i\left(\vec{y},\vec{z},t\right)$, $i=\overline{1,m}$, and their derivatives with respect to $\vec{y}$ and $\vec{z}$ are assumed to be continuous considered in the domain in which their arguments vary.

The purpose of this paper is to determine the conditions under which the order *s* of the initial model may be decreased – replacing it by an approximate (reduced) model of order *n*, which can be obtained from (1) when $\mu_i=0$ for all $i=\overline{1,m}$.

II. Simplification of a Two-timescale System

Initially, we consider *a two-time-scale system*. This is described by model (1) when $i=1$:

$$\begin{cases} \dfrac{d\vec{y}}{dt} = f\left(\vec{y},z,t\right), \\ \mu \dfrac{dz}{dt} = F\left(\vec{y},z,t\right). \end{cases} \tag{2}$$

The possibility of transforming this, as $\mu\to 0$, to *the reduced model*

$$\begin{cases} \dfrac{d\bar{\vec{y}}}{dt} = f\left(\bar{\vec{y}},\bar{z},t\right), \\ 0 = F\left(\bar{\vec{y}},\bar{z},t\right), \end{cases} \tag{3}$$

is considered in the familiar *Tikhonov's theorem on the passage to the limit* [2] (or [3] in English). This gives sufficient conditions under which the solution $\left(\vec{y},z\right)$ of the initial system converges to the solution $\left(\bar{\vec{y}},\bar{z}\right)$ of the reduced system as $\mu\to 0$. We consider these conditions. Model (2) is considered as the complex of models of two subsystems describing the processes with fast and slow speeds of free components of motions. The subsystem of the slow variables is described in (2) by the first vector equation for $\vec{y}$; the subsystem of the fast variables is described by the second scalar equation for *z*. The conditions of Tikhonov's theorem concern demands on the properties of the fast subsystem.

We set the left part of the second equation in (2) to zero

$$0 = F\left(\vec{y}, z, t\right) \tag{4}$$

and solve this equation with respect to *z*. This gives the root of equation (4)

$$z^0 = \varphi\left(\vec{y}, t\right). \tag{5}$$

This contributes to the construction of the reduced system in its final form

$$\frac{d\bar{\vec{y}}}{dt} = f\left(\bar{\vec{y}}, \varphi\left(\bar{\vec{y}}, t\right), t\right). \tag{6}$$

In the model of the fast motions, we transfer to the new independent variable $\tau = {t}/{\mu}$

$$\frac{d\tilde{z}}{d\tau} = F\left(\dot{y}, \tilde{z}, t\right), \tag{7}$$

where $\tilde{z}(\tau) = z\left({t}/{\mu}\right)$.

Equation (7), where $\vec{y}$ and *t* are considered as parameters, is called *the associated system.* Variables $\vec{y}(t)$ and *t* can be treated as constants because their changes are small (when μ is small) in the time intervals where solutions of (7) are investigated for the fulfillment of the conditions of Tikhonov's theorem. For system (7), the root (5) (when $\vec{y}$, *t – const*) is the equilibrium point.

According to [2], in order to make the solution of the initial system (2) tend towards the solution of the reduced system (6), as $\mu \to 0$, two conditions sufficient:

1) The root (5) must be the asimptotically stable equilibrium point of the associated system (7);

2) The initial value $z_0 = z\left(t_0\right)$, when $\vec{y}_0 = \vec{y}(t_0)$, $t = t_0 = 0$, must belong to the domain of influence of the root (5).

We now consider each condition.

The fulfillment of the first condition may be confirmed by means of Lyapunov's first method on the basis of the transfer from (7) to the equation of the first approximation by means of the Taylor expansion of $F\left(\vec{y}, \tilde{z}, t\right)$ at the point z^0

$$\frac{d\Delta\tilde{z}}{d\tau} = \left.\frac{\partial F\left(\vec{y}, \tilde{z}, t\right)}{\partial \tilde{z}}\right|_{\tilde{z}=z^0} \cdot \Delta\tilde{z}. \tag{8}$$

Two cases are possible here [4].

If $\left.\frac{\partial F\left(\vec{y}, \tilde{z}, t\right)}{\partial \tilde{z}}\right|_{\tilde{z}=z^0} \neq 0$ then the necessary and sufficient condition of the asymptotic stability of the root (5) is the demand that

$$\left.\frac{\partial F\left(\vec{y}, \tilde{z}, t\right)}{\partial \tilde{z}}\right|_{\tilde{z}=z^0} < 0. \tag{9}$$

Let $\left.\frac{\partial F\left(\vec{y}, \tilde{z}, t\right)}{\partial \tilde{z}}\right|_{z=z^0} = 0$ and $\left.\frac{\partial^r F\left(\vec{y}, \tilde{z}, t\right)}{\partial \tilde{z}^r}\right|_{\tilde{z}=z^0} \cdot \Delta\tilde{z}^r$ be the first non-zero term of the Taylor expansion of the first part of the equation (7). In this case the asymptotic stability of the root (5) may be so if and only if the following condition is fulfilled:

$$\left.\frac{\partial^r F\left(\vec{y}, \tilde{z}, t\right)}{\partial \tilde{z}^r}\right|_{\tilde{z}=z^0} < 0 \text{, } r \text{ – an odd number.} \tag{10}$$

Inequalities (9) and (10) guarantee the stability of the root (5) when the deviations of *z* from it are small.

The second condition of Tikhonov's theorem demands that the initial value $\tilde{z}_0 = z_0$ must be in the domain of influence of the stable root. The fulfillment of this condition guarantees that the process of the associated system $\tilde{z}$ will reach the stationary point when a deviation from it is determined by the initial conditions.

In the general case equation (4) can have several roots, $z^{0\nu}$, $\nu = \overline{1,k}$, with different stability properties. We can prove the following assertion:

Let equation (4) have k roots, $z^{0\nu}$*, arranged in ascending order, so that* $z^{0(\nu+1)} > z^{0\nu}$.

Then stable roots alternate with unstable roots.

We can show that there is an unstable root $z^{0(\nu+1)}$ between the two nearest stable roots $z^{0\nu}$ and $z^{0(\nu+2)}$. According to (9) or (10), in some neighborhood of the root $z^{0\nu}$ when $z > z^{0\nu}$, the variable $z\left(t\right)$ decreases over time and approaches $z^{0\nu}$, that is: $\frac{dz}{d\tau} < 0$. From (7) this means that $F < 0$. In some neighborhood of the root $z^{0(\nu+2)}$, when $z <^{0(\nu+2)}$, the value of *z* increases, i.e. $\frac{dz}{d\tau} > 0$ and consequently $F > 0$. Thus in the interval $\left(z^{0\nu}, z^{0(\nu+2)}\right)$ the function $F(z)$ changes its sign.

Because the function is continuous this means that a point $z = z^*$ exists, such that, at this point, $F(z^*) = 0$ and the same value is the root of the equation (4). Consider the behavior of $z(t)$ near the root z^*. If $z < z^*$ we get $F < 0$, i.e. $\frac{dz}{d\tau} < 0$ and, consequently, z decreases moving away from z^*. If $z > z^*$ $F > 0$, $\frac{dz}{d\tau} > 0$ and z increases and moves away from z^* as well. From equation (8), derived from (7) by means of expansion of F in z^*, it follows that such behavior of z in the neighborhood of z^* takes place if the derivatives in inequalities (9) or (10) are positive. Thus z^* is an unstable root. It is easy to prove, in the same manner, that between two neighboring unstable roots there is a stable root. Now we show that there is only one unstable root between two neighboring stable roots and that the stable root that lies between two unstable is also unique. Let us consider two neighboring stable roots and suppose that there is more than one unstable root between them. Then, as a result of the statements proved above, there should be stable roots between them. It contradicts the initial hypothesis that we consider the interval of z values between two neighboring stable roots. In the same way it can be proved that the stable root between two neighboring unstable roots is unique. So the assertion is proved.

From this assertion it follows that *the domain of influence of the stable root is the initial values of* z_0 *which are in the interval limited by the values of the neighboring greater and lesser unstable roots*. If the stable root is minimal then its domain of influence enlarges upon values, $z_0 = \left(-\infty, z^{02}\right)$. If the root is maximum then processes converge to it when $z_0 = \left(z^{0(k-1)}, +\infty\right)$.

The proved assertion provides the final answer regarding the fulfillment of the second condition of the theorem. The interval D of the initial condition values, providing the inclusion of the chosen initial value of z_0 in the domain of influence of some stable root, depends on the stability properties of minimal and maximal roots z^{01} and z^{0k}. If z^{01} and z^{0k} are both stable roots then $D = (-\infty, +\infty)$; if z^{01} is a stable root and z^{0k} is an unstable root then $D = \left(-\infty, z^{0k}\right)$; if z^{01} is an unstable root and z^{0k} is a stable root then $D = \left(z^{01}, +\infty\right)$; if z^{01} and z^{0k} are both unstable roots then $D = \left(z^{01}, z^{0k}\right)$. The second condition of the theorem will be fulfilled if the dynamic system is being considered with the initial condition of z_0 belonging to one of the aforementioned intervals D. In this case the very stable root in the domain of influence of which the initial condition z_0 lies will enter into the expression for the reduced system (6). Otherwise the condition of the theorem will not be fulfilled.

III. A PROCEDURE FOR MULTI-TIMESCALE SYSTEM ORDER REDUCTION

We now consider the initial problem of the construction of a reduced model for system (1). A general algorithm for its solution is given in [2] ([5] in English). This assumes a procedure of sequentially decreasing the reduced system dimension from the *(s - 1)*^*th*^ to the *n*^*th*^ value. In each step of the reduction, the equation for z_i, with parameter μ_i, having the largest order of magnitude is considered to be the equation of the fast subsystem. Other equations for z_i together with the equations for $\vec{y}$ comprise the subsystem of the slow variables. The problem of the construction of the reduced model at this step of decomposition is solved in a similar way to the process described above from the initial model (2) to the reduced model (6). The resultant reduced model, after reducing from it the equation with μ_i of the largest order of magnitude, is considered as the initial one for the following step of its order reduction as the process repeats. At each step, the fulfillment of the conditions of the theorem [2] is checked for the associated system obtained during this step, as was described above.

Thus, at the first step, the system (1) is considered as the initial model. Its fast subsystem is determined by the equation with μ_m. The remaining equations for $\vec{y}$ and z_i ($i = \overline{1, m-1}$) refer to the subsystem of the slow variables.

The roots are obtained from the equation

$$0 = F_m\left(\vec{y}, \vec{z}, t\right),$$

and the associated system is the following:

$$\frac{d\tilde{z}_m}{d\tau_1} = F_m\left(\vec{y}, z_1, \ldots, z_{m-1}, \tilde{z}_m, t\right), \tag{11}$$

where $\vec{y}$, z_i ($i = \overline{1, m-1}$), t are parameters determined by their initial values, $\tau_1 = {t}/{\mu_1}$. The fulfillment of the conditions of the theorem [2] is checked when the initial condition of the solution $z_m\left(t_0\right) = z_{m0}$. If they are fulfilled, we chose the root, $z_m^0 = \varphi_m\left(\vec{y}, z_1, \ldots, z_{m-1}, t\right)$, in which the domain of influence of the given initial condition, z_{m0}, lies. As a result, we obtain the reduced system of the first step:

$$\begin{cases} \dfrac{d\vec{y}}{dt} = f\left(\vec{y}, z_1, \ldots, z_{m-1}, \varphi_m\left(\vec{y}, z_1, \ldots, z_{m-1}, t\right), t\right), \\ \mu_i \dfrac{dz_i}{dt} = F_i\left(\vec{y}, z_1, \ldots, z_{m-1}, \varphi_m\left(\vec{y}, z_1, \ldots, z_{m-1}, t\right), t\right), \end{cases} \tag{12}$$

$i = \overline{1, m-1}.$

At the second step, the initial system is the system (12), where the fast subsystem is described by the previous equation with μ_{m-1} and so on. For the all associated systems received in steps 1 … *m*, the fulfillment of conditions (1) and (2) is checked. In this case, the associated system received in the k^{th} step is the following:

$$\frac{d\tilde{z}_j}{d\tau_k} = F_j\left(\vec{y}, z_1, \ldots, z_{j-1}, \tilde{z}_j, \varphi_{j+1}(\vec{y}, z_1, \ldots, z_{j-1}, \tilde{z}_j, t),\right.$$

$$\varphi_{j+2}(\vec{y}, z_1, \ldots, z_{j-1}, \tilde{z}_j, \varphi_{j+1}(\cdot), t), \ldots, \tag{13}$$

$$\left.\varphi_m(\vec{y}, z_1, \ldots, z_{j-1}, \tilde{z}_j, \varphi_{j+1}(\cdot), \ldots, \varphi_{m-1}(\cdot), t), t\right), \; j = m - k + 1,$$

where all variables on the right-hand side, except $\tilde{z}_j$, are considered as parameters, $\tau_k = t/\mu_k$, functions F_j depend on t, $\vec{y}$, z_1, ..., z_{j-1}, $\tilde{z}_j$ and m - j functions $\varphi_{j+1}(\cdot)$, ..., $\varphi_m(\cdot)$. (Here, for brevity, we do not write down the dependence of all functions on their variables: replacing it by " $\cdot$ " where this dependence is unambiguous.)

If the conditions described above are satisfied for all associated systems (i.e. for systems (13), when $k = \overline{1,m}$), then the following reduced model corresponds to the initial model (1):

$$\frac{d\bar{\vec{y}}}{dt} = f\left(\bar{\vec{y}}, \varphi_1(\cdot), \ldots, \varphi_m(\cdot), t\right), \tag{14}$$

where $\varphi_i(\cdot)$, $i = \overline{1,m}$, are the chosen roots.

IV. Conclusion

In conclusion, let us represent the results obtained as a procedure for finding the reduced model for model (1), requiring no special knowledge of singularly perturbed systems of differential equations.

In system (1) let us distinguish the equation for z_m. From equation $F_m\left(\vec{y}, z_1, \ldots, z_{m-1}, z_m, t\right) = 0$, we obtain roots $z_m^{0\nu} = \varphi_m^{\nu}\left(\vec{y}, z_1, \ldots, z_{m-1}, t\right)$, $\nu = \overline{1, k_m}$. For each of these, the stability or instability properties for small deviations are investigated. The root is stable if one of the inequalities (9) or (10) holds for it; otherwise it is unstable. In inequalities (9) and (10) the partial derivatives of function F_m are considered at points $z_m^{0\nu}$. According to the rule given above, the limits of the interval *D* and its division into domains of stable root influence are determined. The belonging of the initial condition z_{m0} to the received interval *D* is checked. If the result of the check is positive, the root z_m^{0l} is refined in the influence domain in which it lies. The nearest to the z_{m0} stable root, which is not separated from the initial condition by the unstable root, is chosen. Substituting this value $z_m^{0l} = \varphi_m^{l}\left(\vec{y}, z_1, \ldots, z_{m-1}, t\right)$ into the initial system (1) we obtain the reduction for the first step of the procedure model with the reduction of its order by one.

The reduced model obtained is then the initial one for the second step of the procedure. In this step, equation $z'_{m-1} = F_{m-1}\left(\vec{y}, z_1, \ldots, z_{m-1}, \varphi_l\left(\vec{y}, z_1, \ldots, z_{m-1}, t\right), t\right)$, and roots $z_m^{0\nu} = \varphi_m^{\nu}\left(\vec{y}, z_1, \ldots, z_{m-1}, t\right)$, $\nu = \overline{1, k_m}$, of equation $F_{m-1} = 0$ are considered.

Let us assume that, at each step of the degeneration of the reduced model order, the presence of the stable root, to the domain of which influence the initial value for this step (for example, for *i*-th step it is $z_{(m-i+1)0}$) belongs, is confirmed. Then, in *m* steps, system (1) will be represented by a system of *n* differential equations with respect to variables $\vec{y}$ and *t*, which represent the description of the completely reduced model corresponding to the initial model with the reduction of its dimension by *m* orders. The description of processes z_1, ..., z_m is found by successive substitution of the known variables in the expressions for stable roots z_i^0 distinguished on each step, beginning with z_1^0.

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